Persistence of a continuous stochastic process with discrete-time sampling: Non-Markov processes

George C. M. A. Ehrhardt,[1] Alan J. Bray,[1] and Satya N. Majumdar[2]
[1]*Department of Physics and Astronomy, University of Manchester, Manchester, M13 9PL, United Kingdom*
[2]*Laboratoire de Physique Quantique (UMR C5626 du CNRS), Université Paul Sabatier, 31062 Toulouse Cedex, France*

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We consider the problem of "discrete-time persistence," which deals with the zero crossings of a continuous stochastic process $X(T)$ measured at discrete times $T=n\Delta T$. For a Gaussian stationary process the persistence (no crossing) probability decays as $\exp(-\theta_D T)=[\rho(a)]^n$ for large n, where $a=\exp(-\Delta T/2)$ and the discrete persistence exponent θ_D is given by $\theta_D=(\ln\rho)/(2\ln a)$. Using the "independent interval approximation," we show how θ_D varies with ΔT for small ΔT and conclude that experimental measurements of persistence for smooth processes, such as diffusion, are less sensitive to the effects of discrete sampling than measurements of a randomly accelerated particle or random walker. We extend the matrix method developed by us previously [Phys. Rev. E **64**, 015101(R) (2001)] to determine $\rho(a)$ for a two-dimensional random walk and the one-dimensional random-acceleration problem. We also consider "alternating persistence," which corresponds to $a<0$, and calculate $\rho(a)$ for this case.

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I. INTRODUCTION

Persistence of a continuous stochastic variable has recently been a subject of considerable interest among both theoreticians and experimentalists. Systems studied include randomly driven single degrees of freedom [1–4], simple diffusion from random initial conditions [5,6], models of phase separation [7–17], fluctuating interfaces [18–20], and reaction-diffusion processes [21–24]. For a recent review see Ref. [25]. Persistence is the probability $P(t)$ that a fluctuating nonequilibrium field, at a particular space point, has not crossed a certain threshold (usually its mean value) up to time t. In most systems studied, scale invariance implies that for large t, P exhibits a power-law decay $P(t)\sim t^{-\theta}$, where the persistence exponent θ is nontrivial due to the dependence of $P(t)$ on the whole history of the system. Experiments have recently measured θ for the coarsening dynamics of breath figures [26], liquid crystals [27], soap bubbles [28], and diffusion of Xe gas in one dimension [29].

In this paper we consider the following problem: in any experimental (or numerical) measurement of θ, the stochastic variable studied, $x(t)$, will have to be sampled discretely. It is, therefore, possible that $x(t)$ could cross and then recross its threshold between samplings, resulting in a false-positive classification of the persistence of $x(t)$. If the sampling is logarithmically spaced in time (as was the case in Ref. [29]), then such undetected crossings will make the measured persistence exponent smaller than theory predicts (while if the sampling is uniform in real time only the prefactor is changed). This problem has been studied in Ref. [30] for the case of a random walker in one dimension, a simple Markovian, "rough" (i.e., with a fractal distribution of crossings [31]) process. Here we extend that work to consider a simple, nonMarkovian, "smooth" process: a randomly accelerated particle. Since most of the more complex processes studied experimentally are "smooth," this paper is a step towards understanding the effect discrete sampling has on those measured persistence exponents.

There is yet another motivation for studying the persistence of a discrete sequence as opposed to that of a continuous process. It turns out that many continuous processes in nature are stationary under translations of time *only* by an integer multiple of a basic period (which can be chosen to be unity without any loss of generality). For example, the weather records have this property due to seasonal repetitions. It has recently been shown [32] that for a wide class of such processes, the continuous time persistence $P(t)$ is the same as the persistence $P(n)$ of the corresponding discrete sequence resulting from the measurement of the continuous data only at discrete integer points. In general, the calculation of the persistence of a discrete sequence is much harder than that of a continuous process, except in special cases where $P(n)$ can be computed exactly [32,33]. The tools that have been developed over the last decade for studying the persistence of a continuous process are often not easily extendable to the case of discrete sequences and one needs to invoke different techniques, some of which are presented in this paper.

The layout of this paper is as follows. In Sec. II we use the independent interval approximation (IIA) [5,6] to find the first correction to θ for small ΔT, where ΔT is the separation of samplings in logarithmic time ($T=\ln t$). This gives us some indication of how significant the effect of discrete sampling is. For example, for the random walk $\theta_D-\theta\propto-\sqrt{\Delta T}$ and so the discrete exponent θ_D and its continuum limit θ begin to deviate markedly as soon as $\Delta T>0$. For the random-acceleration problem, one finds $\theta_D-\theta\propto-\Delta T$, a weaker dependence on ΔT, while for the diffusion equation from random initial conditions, $\theta_D-\theta\propto-(\Delta T)^2$, so the effect of discrete sampling for small ΔT is much less significant in this case. These three systems display an increasing "smoothness" of the underlying process, a concept we will expand upon below.

In Sec. III we illustrate our general method for solving low-dimensional discrete persistence problems by considering the case of a random walker in one dimension, $\dot{x}(t)=\eta(t)$, where $\eta(t)$ is Gaussian white noise. A brief account of this work was given in Ref. [30]. We map the problem to a Gaussian stationary process (GSP) in the variable $X = x/\sqrt{t}$ by transforming to logarithmic time. In the new variables, the random walk is represented by an Ornstein-Uhlenbeck process. We repose this problem in terms of a backward Fokker-Plank equation (BFPE) and give the solution for the continuum case where the persistence exponent has the well-known value $\theta = 1/2$. We then formulate the discrete persistence problem in terms of an eigenvalue integral equation. Employing a power series expansion of the integrand reduces the problem to a matrix eigenvalue equation, whose largest eigenvalue gives us the persistence exponent measured by discrete sampling. The concept of alternating persistence, in which the consecutive measured values of X lie on alternate sides of the threshold value is also introduced and θ_D is calculated for this case.

Having demonstrated the matrix method on a single variable problem, we then apply it to two-variable problems: the two-dimensional random walk $\dot{x}(t) = \eta(t)$, in a wedge geometry (Sec. IV), and a randomly accelerated particle in one dimension, $\ddot{x}(t) = \eta(t)$, which is a simple example of a non-Markovian process (Sec. V). Surprisingly, for alternating persistence in the continuum limit ($\Delta T = 0$) the asymptotic probability of surviving one further sampling, ρ, is nonzero, a phenomenon contrary to our earlier suggestion [30] that this should only occur for "rough" processes. Using scaled variables and logarithmic time, the random-acceleration problem maps onto a damped simple-harmonic oscillator, which we study for the overdamped case using the matrix method, finding θ_D as a function of ΔT. Using the results of a correlator expansion developed in Ref. [34] we also study the underdamped case, an example for which the correlator is oscillatory (Sec. VI). We show that when the time interval ΔT between measurements is equal to the period of the oscillation, the problem is identical to the Ornstein-Uhlenbeck process studied in Sec. III, while for ΔT equal to one-half of a period the problem reduces to that of alternating persistence in the Ornstein-Uhlenbeck process. The paper concludes with a summary of the results.

II. SMALL ΔT CORRECTION TO θ

The independent interval approximation [5,6] uses the assumption that the intervals between zero crossings of a GSP are independently distributed. Although this assumption is not valid for most processes, it nevertheless gives remarkably accurate estimates for θ in many cases. Here we use the same assumption to find the first correction to θ due to discrete sampling with a spacing in logarithmic time of ΔT for ΔT small.

As ΔT is increased from zero, the first correction to θ comes from paths that are always positive apart from one undetected double crossing between consecutive sample times. Let the probability of one such double crossing occurring in the interval $T = n\Delta T$ be $P_{dble}(n,\Delta T)$. Then, for T large, the probability $P_D(T)$ that the stochastic variable is positive at all n samplings is given, to lowest order in ΔT, by

$$P_D(T) = P_0(T) + P_{dble}(n,\Delta T), \tag{1}$$

where $P_0(T)$ is the continuous-time persistence probability, given by $P_0(T) \sim e^{-\theta T}$ for T large, and $P_{dble}(n,\Delta T)$ is given by

$$P_{dble}(n,\Delta T) \propto n e^{-\theta n \Delta T} \int_0^{\Delta T} dT_1 \int_{T_1}^{\Delta T} dT_2 P_1(T_2 - T_1), \tag{2}$$

where $P_1(T)$ is the probability distribution of the interval size, and we have assumed that the durations of different intervals are statistically independent. The latter assumption is precisely the IIA. It is clear from Eq. (2) that, to leading order in ΔT, we only require the form of $P_1(T)$ in the limit $T \to 0$. The function $P_1(T)$ can be found using the IIA and for the processes currently under consideration—the random walk, random acceleration, and diffusion from random initial conditions—the small-T results are $P_1(T) \propto 1/\sqrt{T}$, 1, and T, respectively. All three cases are incorporated in the general form $P_1(T) \propto T^\alpha$, with $\alpha = -1/2$, 0, and 1, respectively. Using this form in Eq. (2) gives

$$P_D(T) = P_0(T) + \gamma n \Delta T^{\alpha+2} e^{-\theta T}, \tag{3}$$

where γ is some constant. Since $T = n\Delta T$ and $P_0(T) \sim e^{-\theta T}$, we have

$$P_D(T) = A e^{-\theta T}(1 + B\, T \Delta T^{\alpha+1}), \tag{4}$$

where A, B are constants, and so, to lowest order in ΔT,

$$P_D(T) = A e^{-\theta T} e^{B\, T \Delta T^{\alpha+1}}. \tag{5}$$

Since $P_D(T) \sim e^{-\theta_D T}$, we obtain

$$\theta_D = \theta - B \Delta T^{\alpha+1}. \tag{6}$$

For the random walk, random acceleration, and diffusion from random initial conditions, $\theta_D - \theta \propto -\sqrt{\Delta T}$, $-\Delta T$, and $-(\Delta T)^2$, respectively. From this we expect that the discrete sampling is important as soon as ΔT is nonzero for the random walk, this being related to the fact that $P_1(T) \to \infty$ for $T \to 0$, i.e., that the distribution of crossings is fractal and hence this process is "rough" [31]. The persistence exponent for the random-acceleration process is linear in ΔT and thus we expect it to be less affected by the discrete sampling for small ΔT . In Secs. III and IV these expectations will be confirmed using the matrix method perturbative expansion in $a = e^{-\Delta T/2}$ about $a = 0$. Finally, for diffusion from random initial conditions, the $(\Delta T)^2$ dependence indicates that discrete sampling will be relatively unimportant for small ΔT. Finally we note that, although Eq. (6) has been derived using the IIA, we expect it to be valid quite generally. In particular, the $\alpha = 2$ result for $P_1(T)$ has been proved correct by Zeitak [37], the IIA even giving the correct coefficient.

III. RANDOM WALK IN ONE DIMENSION

Let us consider the simple case of a random walker in one dimension,

$$\dot{x}(t) = \eta(t), \tag{7}$$

where $\eta(t)$ is Gaussian white noise with zero mean and $\langle \eta(t)\eta(t')\rangle = 2D\delta(t-t')$. For convenience we map this process, which is nonstationary in time, to a Gaussian stationary process by defining new space and time variables $X = x/\sqrt{t}$ and $T=\ln t$. Equation (7) then reads

$$\frac{dX}{dT} = -\mu X(T) + \eta(T), \tag{8}$$

where $\langle \eta(T)\eta(T')\rangle = 2D\delta(T-T')$ and $\mu = 1/2$ for the random walker, although other values of μ can be considered. This Ornstein-Uhlenbeck problem can be more usefully solved using the BFPE. Let $Q(X,T)$ be the probability that the random walker, starting from X at time $T=0$, does not cross the origin ($X=0$) up to time T. Then from Eq. (8) it can be shown that $Q(X,T)$ satisfies

$$\partial Q/\partial T = D\partial^2 Q/\partial X^2 - \mu X \partial Q/\partial X, \tag{9}$$

with initial condition $Q(X,0)=1$ for all $X>0$ and boundary conditions $Q(0,T)=0$ and $Q(\infty,T)=1$. The solution is

$$Q(X,T) = \mathrm{erf}\left[\frac{e^{-\mu T}}{\sqrt{2D'(1-e^{-2\mu T})}} X\right], \tag{10}$$

where erf(x) is the error function and $D' = D/\mu$. For large T (and positive μ), $P(T) \sim X e^{-\mu T}$, which corresponds to $P(t) \sim t^{-\theta}$ in real time, with $\theta = \mu$.

We now consider discrete persistence, i.e., the probability $Q_n(X)$ that starting at X our field/variable is positive at all the discrete sample times $T_1 = \Delta T$, $T_2 = 2\Delta T, \ldots, T_n = n\Delta T$. This is relevant for experimental or numerical determinations of θ since in practice one will have to sample only at discrete points. In Ref. [29], for example, the sampling is done logarithmically in real time (which corresponds to sampling uniformly in logarithmic time). Note that θ_D will, in general, be smaller than the continuum value θ since any even number of crossings between samplings will go unnoticed. One can write down a recurrence relation for $Q_n(X)$,

$$Q_{n+1}(X) = \int_0^\infty dY Q_n(Y) P(Y,\Delta T|X,0), \tag{11}$$

where $P(Y,\Delta T|X,0)$ is the Greens function, i.e., the probability that a particle starting at X at time zero will be at Y at time ΔT. For a Gaussian process, $P(Y,\Delta T|X,0)$ can be found from the mean and variance of $X(T)$,

$$P(Y,\Delta T|X,0) = \frac{1}{\sqrt{2\pi D'(1-a^2)}} \exp\left[-\frac{(Y-aX)^2}{2D'(1-a^2)}\right], \tag{12}$$

where $a = \exp(-\mu\Delta T)$. This gives us the discrete analogue of the BFPE (9),

$$Q_{n+1}(X) = \int_0^\infty dY\, Q_n(Y) \frac{1}{\sqrt{2\pi D'(1-a^2)}} \times \exp\left[-\frac{(Y-aX)^2}{2D'(1-a^2)}\right], \tag{13}$$

with the continuum equation being recovered in the limit $\Delta T \to 0$.

Making the change of variables $x = X/\sqrt{D'(1-a^2)}$, $y = Y/\sqrt{D'(1-a^2)}$, and $Q_n(X) = Q_n'(x)$ gives

$$Q_{n+1}'(x) = \frac{1}{\sqrt{2\pi}} \int_0^\infty dy\, Q_n'(y) \exp[-(y-ax)^2/2]. \tag{14}$$

At late times, with $\mu > 0$, we expect $Q_{n+1}'(x) = \rho Q_n'(x)$ in analogy with the continuous case where $P(T+\Delta T) = P(T) e^{-\theta\Delta T}$ so $\rho = e^{-\theta_D \Delta T}$. We, therefore, expect that, for large n, $Q_n(x) \to \rho^n q(x)$. Substituting this into Eq. (14) gives an eigenvalue integral equation for $q(x)$

$$\rho q(x) = \frac{1}{\sqrt{2\pi}} \int_0^\infty dy\, q(y) \exp[-(y-ax)^2/2], \tag{15}$$

with an eigenvalue $\rho(a)$ that depends continuously on a. Equation (15) has an infinite number of eigenvalues, but at late times only the largest will remain since $\Sigma_i \rho_i^n \approx \rho_{max}^n$ for large n (there is no other eigenvalue contiguous to the largest eigenvalue). By symmetrizing the kernel in Eq. (15), one can use the variational method to find a rigorous lower bound for ρ [30]. This method cannot be applied to the random-acceleration problem, however, since the kernel cannot be symmetrized. For $\Delta T \to \infty$, correlations between the nth and $(n+2)$th samplings are negligible, so

$$P_n \approx [\text{Prob (two consecutive points have the same sign)}]^n \tag{16}$$

$$= \left(\frac{1}{2} + \frac{1}{2}\langle \mathrm{sgn}[X(0)]\mathrm{sgn}[X(\Delta T)]\rangle\right)^n \tag{17}$$

$$= \left(\frac{1}{2} + \frac{1}{2}\frac{2}{\pi}\arcsin[C(\Delta T)]\right)^n, \tag{18}$$

where $C(T_2 - T_1) = \exp[-\mu(T_2 - T_1)]$ is the normalized autocorrelation function of $X(T)$, i.e., $C(T) = \langle X(T)X(0)\rangle / \langle X^2\rangle$. Hence, for large ΔT,

$$\rho = \frac{1}{2} + \frac{a}{\pi} + \cdots. \tag{19}$$

Equation (15) can also be solved perturbatively by expanding the exp(axy) term as a power series in a

$$\rho q(x) = \frac{1}{\sqrt{2\pi}} \int_0^\infty dy q(y) e^{-y^2/2} e^{-a^2x^2/2} \sum_{m=0}^\infty \frac{(axy)^m}{m!}. \tag{20}$$

Defining

$$b_m = \frac{a^{m/2}}{\sqrt{m!}} \int_0^\infty dy\, q(y) y^m e^{-y^2/2} \tag{21}$$

gives

$$\rho q(x) = \frac{1}{\sqrt{2\pi}} e^{-a^2x^2/2} \sum_{m=0}^{\infty} \frac{b_m}{\sqrt{m!}} (\sqrt{a}x)^m. \tag{22}$$

Multiplying through by $e^{-x^2/2}x^n$ and integrating over $x>0$ gives

$$\rho b_n = \sum_{m=0}^{\infty} A_{nm} b_m \tag{23}$$

where

$$A_{nm} = \frac{1}{\sqrt{4\pi(1+a^2)}} \left(\frac{2a}{1+a^2}\right)^{(n+m)/2} \frac{\Gamma[(n+m+1)/2]}{\sqrt{n!m!}}. \tag{24}$$

By this method, the eigenvalue integral equation (15) has been converted to the eigenvalue matrix equation (23), and the problem reduces to computing the largest eigenvalue of an $N\times N$ submatrix whose (n,m)th elements decrease exponentially in $n+m$. In [30] we determined ρ numerically to one part in 10^{12}. We have also algebraically found ρ_{max} as a series expansion in a, the first four terms being

$$\rho = \frac{1}{2} + \frac{a}{\pi} + \frac{\pi-2}{\pi^2}a^2 + \frac{48-36\pi+7\pi^2}{6\pi^3}a^3 + \cdots. \tag{25}$$

The coefficients up to order a^{49} are given in Appendix B. For $\Delta T\to 0$, $a\to 1$, and convergence becomes progressively slower. However, the variational method still works in this region.

One may also consider the case of alternating persistence, i.e., the probability that $X(n\Delta T)$ is positive for every even n and negative for every odd n (or vice versa). The limit of integration in Eq. (15) then changes, giving

$$\rho q(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^0 dy q(y) \exp[-(y-ax)^2/2]. \tag{26}$$

Substituting $y\to -y$, swapping the limits of integration and using $q(-y)=q(y)$ (since the process is symmetric around $y=0$) gives

$$\rho q(x) = \frac{1}{\sqrt{2\pi}} \int_0^\infty dy q(y) \exp[-(y+ax)^2/2], \tag{27}$$

which is identical to Eq. (15), but with a replaced by $-a$. Therefore, replacing a by $-a$ in the matrix equation (23) will give the alternating persistence eigenvalue ρ_{alt}. The case $|a|>1$ may also be considered, this corresponding to $\mu<0$ and hence to an unstable potential in the Ornstein-Uhlenbeck process. For the alternating case, $a<-1$, the calculation proceeds as before. In fact, from Eq. (24) it can be seen that $\rho(1/a)=\rho(a)|a|$ for $a<1$, so one need only investigate alternating persistence in the range $-1<a<0$. For $a>1$ it is possible for the walker to escape to infinity, i.e., $q_n(x)\not\to 0$ for $n\to\infty$. The asymptotic limit for $q(x)$ when $a>1$ may be found using the matrix method in the following way. Since $q(x)\to 1$ as $x\to\infty$, it is more convenient to study a new function $u(x)$ defined by the relation

$$q(x) = 1 - q(0)\int_x^\infty u(y)dy, \tag{28}$$

where $q(0)$ is fixed by Eq. (28) with $x=0$. Substituting $q(x)$ from Eq. (28) into Eq. (15) (with $\rho=1$ since we are finding the stationary state) we find, after some algebra, that $u(x)$ satisfies the integral equation

$$u(x) = \frac{a}{\sqrt{2\pi}}\left[e^{-a^2x^2/2} + \int_0^\infty u(y)\exp[-(y-ax)^2/2]dy\right]. \tag{29}$$

Note that, unlike Eq. (15), which determines $q(x)$ only up to an overall multiplicative constant, Eq. (29) is an inhomogeneous equation that fixes $u(x)$ absolutely, as one expects on physical grounds.

As before, we expand the factor exp(axy) in Eq. (29) as a power series to obtain

$$u(x) = \frac{a}{\sqrt{2\pi}} e^{-a^2x^2/2}\left[1 + \sum_{n=0}^{\infty} \frac{a^n x^n}{n!} \int_0^\infty dy y^n u(y) e^{-y^2/2}\right]. \tag{30}$$

Multiplying through by $x^m(a^{m/2}/\sqrt{m!})e^{-x^2/2}$, integrating over positive x, and defining

$$c_n = \frac{a^{n/2}}{\sqrt{n!}} \int_0^\infty dy\, y^n u(y)\, e^{-y^2/2}, \tag{31}$$

gives

$$c_n = a\left[A_{n0} + \sum_{m=0}^{\infty} A_{nm}c_m\right], \tag{32}$$

where A_{nm} is our previous matrix given by Eq. (24), and

$$u(x) = \frac{a}{\sqrt{2\pi}} e^{-a^2x^2/2}\left[1 + \sum_{n=0}^{\infty} \frac{a^{n/2}x^n}{\sqrt{n!}} c_n\right]. \tag{33}$$

Equation (32) can be solved by matrix inversion

$$c_n = a\sum_m (B^{-1})_{nm} A_{m0}, \tag{34}$$

where

$$B_{nm} = \delta_{nm} - aA_{nm}. \tag{35}$$

The solution converges rapidly as a function of the size N of the matrix. In practice, N of order a few hundred gives very precise results.

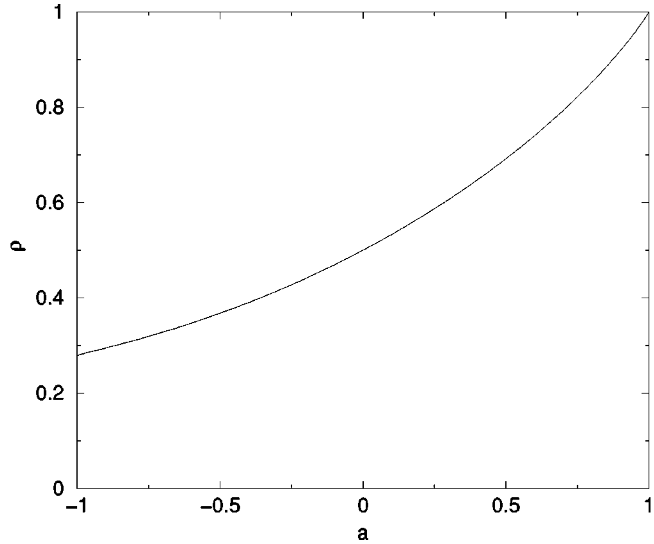

FIG. 1. Plot of $\rho(a)$ vs $a(a=e^{-\mu\Delta T})$, where $a<0$ corresponds to alternating persistence.

Figure 1 shows $\rho(a)$ for both alternating and normal persistence, while Fig. 2 shows $\theta(a)$. Note that even for $a=0.96$, θ_D is significantly below the continuum value of 1/2. We argued in Sec. II that the difference $\theta(1)-\theta(a)$ decreases at $\Delta T^{1/2}$, i.e., as $(1-a)^{1/2}$ for $a\to 1$, implying a square-root cusp at $a=1$ in Fig. 2. Wong *et al.* [29], measuring persistence in one-dimensional diffusion of Xe gas, sampled logarithmically in time such that in log time their ΔT was about 0.24. If the process were a random walk, this would give $a\sim 0.9$ and a difference between θ_D and the continuum θ of about 20%. For the diffusion equation that describes the experiment, however, the approach to the continuum is more rapid (see the discussion in Sec. II), and rather accurate results are obtained even for $\Delta T=0.24$ [34].

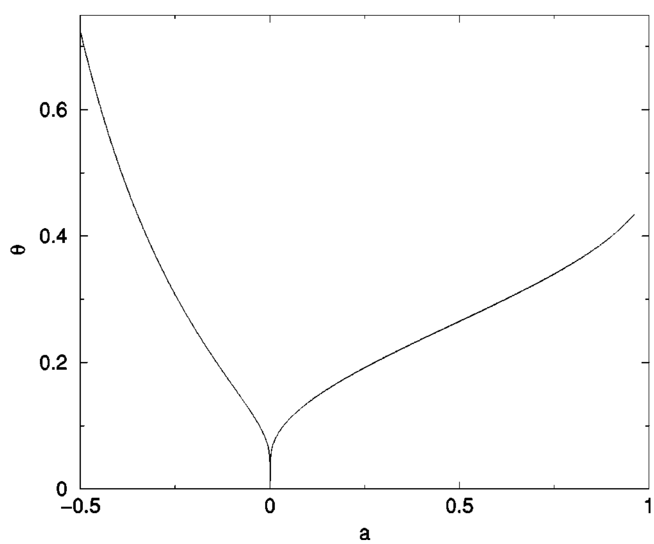

FIG. 2. Plot of $\theta(a)$ vs $a(a=e^{-\mu\Delta T})$, where $\theta=\ln[\rho(a)]/(2\ln|a|)$ and $a<0$ corresponds to alternating persistence. For $a\to 1$, the series has not yet converged as the $1/\ln(a)$ term amplifies the small numerical error in $\rho(a)$, while for $a\to -1$, $\theta\sim 1/\ln(|a|)$.

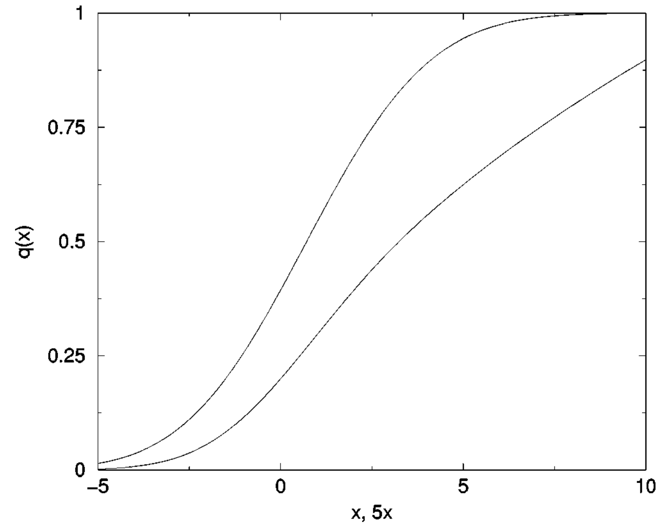

FIG. 3. Plot of eigenfunctions $q(x)$ for $a=0.5$ (lower curve) and $a=2.0$ (upper curve, abscissa$=5x$).

In Fig. 3 we show the eigenfunction $q(x)$ for the case $a=0.5$. This function was obtained by substituting the eigenvector of the matrix A corresponding to the largest eigenvalue into Eq. (22). The asymptotic behavior for large x is $q(x)\sim x^{\nu}$, where $\nu=\ln\rho/\ln a\simeq 0.530\,661$ for $a=0.5$ [30]. Figure 3 also contains a plot of $q(x)$, given by Eq. (28), for the case $a=2$, which corresponds to an unstable potential.

IV. RANDOM WALK IN TWO DIMENSIONS: WEDGE GEOMETRY

Having illustrated the perturbative method on a simple Markovian case for which another approach (the variational method) is available, we intend to study a simple "smooth" non-Markovian process, the random-acceleration problem, $\ddot{x}=\eta(t)$. This process is equivalent to a Markovian problem in two variables $x(t)$ and $v(t)$, where $\dot{v}=\eta(t)$ and $\dot{x}=v$. Before dealing with this problem, we will first consider another two-variable Markov process, namely, the random walk in two dimensions, $\dot{x}=\eta_x(t)$, $\dot{y}=\eta_y(t)$, with $\langle\eta_i(t)\rangle=0$ and $\langle\eta_i(t)\eta_j(t')\rangle=2D\delta_{ij}\delta(t-t')$. Using the perturbative approach on this pedagogical problem will clarify its use on the random-acceleration problem.

Consider a wedge of angle α whose boundaries are absorbing, let our random walker start inside this wedge at radial position r and angle ϕ, with $0\leqslant\phi\leqslant\alpha$. Making the change of variable $R=r/\sqrt{t}$ and $T=\ln t$, converts the problem into a GSP, as in Sec. III. The corresponding BFPE, i.e., the two-dimensional analog of Eq. (9), is

$$\partial Q/\partial T=\nabla^2 Q-\mu R\,\partial Q/\partial R, \qquad (36)$$

where $\mu=1/2$ for the random walk, though we will treat μ as arbitrary, and $Q(R,\phi,T)$ is the survival probability of the particle at time T given that it started at (R,ϕ). The initial condition is $Q(R,\phi,0)=1$ for $R>0$ and $0<\phi<\alpha$, while the boundary conditions are $Q(R,0,T)=0$, $Q(R,\alpha,T)=0$, $Q(0,\phi,T)=0$, and $Q(\infty,\phi,T)=1$ for $0<\phi<\alpha$. The solution can be obtained using separation of variables. Its asymptotic form for $T\to\infty$ at fixed R and ϕ is

$$Q(R,\phi,T)\propto R^{\pi/\alpha}\sin(\pi\phi/\alpha)\exp(-\mu\pi T/\alpha), \tag{37}$$

giving $\theta=\mu\pi/\alpha$. Now consider the discrete persistence. The analog of Eq. (15) is

$$\rho q(\mathbf{r})=\frac{1}{2\pi}\int d\mathbf{r}'\,q(\mathbf{r}')\exp[-(\mathbf{r}'-a\mathbf{r})^2/2], \tag{38}$$

where $a=\exp(-\mu\Delta T)$ as before, $\mathbf{r}=(1-a^2)^{-1/2}\mathbf{R}$, and the integration is over the wedge. Note that for $\alpha=\pi$ one can do the integration over x and recover the one-dimensional result. In polar coordinates, Eq. (38) becomes

$$\rho q(r,\phi)=\frac{1}{2\pi}\int_0^\infty r'dr'\int_0^\alpha d\phi'\,q(r',\phi')\exp[-\{r'^2+a^2r^2 -2arr'\cos(\phi-\phi')\}/2]. \tag{39}$$

As before, we expand the exponential term containing the mixed terms

$$\exp[arr'\cos(\phi-\phi')] =\sum_{m',n'}a^{m'+n'}(rr')^{m'+n'}[\cos(\phi)\cos(\phi')]^{m'} \times[\sin(\phi)\sin(\phi')]^{n'}/[m'!n'!], \tag{40}$$

and define

$$b_{m'n'}=[a^{(m'+n')/2}/\sqrt{m'!n'!}]\int_0^\alpha d\phi' \times\int_0^\infty dr'\,r'^{m'+n'+1}e^{-r'^2/2}[\cos^{m'}(\phi')\sin^{n'}(\phi')] \times q(r',\phi'), \tag{41}$$

giving

$$\rho q(r,\phi)=[e^{-a^2r^2/2}/(2\pi)]\sum_{m',n'}r^{m'+n'}a^{(m'+n')/2} \times[\cos^{m'}(\phi)\sin^{n'}(\phi)]b_{m'n'}/\sqrt{m'!n'!}. \tag{42}$$

Multiplying throughout by

$$(a^{(m+n)/2}/\sqrt{m!n!})r^{m+n+1}e^{-r^2/2}\cos^m(\phi)\sin^n(\phi), \tag{43}$$

and integrating over the wedge, gives

$$\rho b_{mn}=\sum A_{mn,m'n'}b_{m'n'}. \tag{44}$$

Note that, by regarding mn as a single index, we can treat $A_{mn,m'n'}$ as a conventional matrix when solving numerically for ρ. The matrix elements are

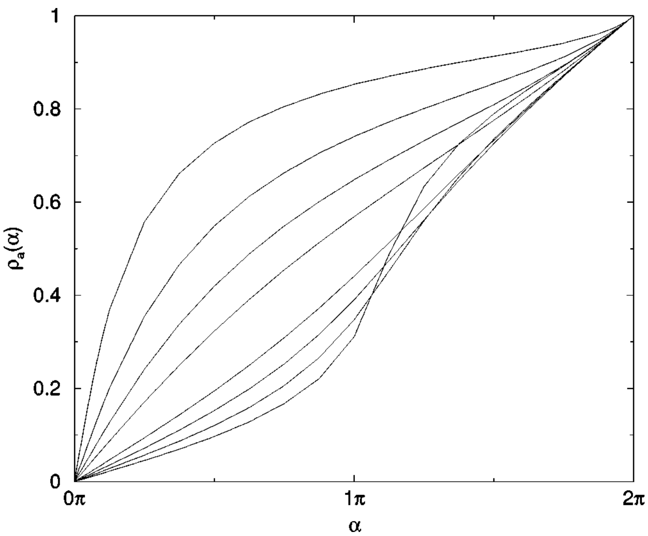

FIG. 4. Plot of $\rho_a(\alpha)$ against $\alpha, a=0.8$ (top curve), 0.6,0.4,0.2, also the alternating cases, $a=-0.2,-0.4,-0.6,-0.8$ (bottom curve). The lines are linear interpolations between the discrete data points.

$$A_{mn,m'n'}=\frac{1}{4\pi a\sqrt{m!n!m'!n'!}}\left[\frac{2a}{a^2+1}\right]^{(m+n+m'+n'+2)/2} \times\Gamma[(m+n+m'+n'+2)/2]J_{mn,m'n'}, \tag{45}$$

where

$$J_{mn,m'n'}=\int_0^\alpha d\phi\,[\cos(\phi)]^{m+m'}[\sin(\phi)]^{n+n'}. \tag{46}$$

The problem has been reduced to an eigenvalue matrix equation in four variables that we solve numerically. We have also found ρ as a series expansion in a. As before we can consider alternating persistence, i.e., the probability that between each sampling the particle alternates between $0<\phi<\alpha$ and $\pi<\phi<\alpha+\pi$. Figure 4 shows a plot of the results. The case $\alpha=\pi$ corresponds to the one-dimensional case while $\alpha=0$ and $\alpha=\pi$ trivially give $\rho=0$ and $\rho=1$, respectively.

For $\Delta T\to\infty$, $\rho\to\alpha/2\pi$ since the system reequilibrates between samplings. In this limit the function $\rho(\alpha)$ is a straight line from (0,0) to $(2\pi,1)$.

In the limit $\alpha\to 0$, we can expand the ϕ' integral in Eq. (39) to first order in α, and set $\phi=0=\phi'$ to obtain

$$\frac{\rho}{\alpha}q(r,0)=\frac{1}{2\pi}\int_0^\infty r'dr'\,q(r',0)\exp[-(r'^2+a^2r^2 -2arr')/2], \tag{47}$$

which is identical to the one-dimensional random walk equation (15) but with $r'dr'$ rather than just dr', and ρ replaced by $\sqrt{2\pi}\rho/\alpha$. This gives a matrix equation similar to that for the $d=1$ case, but with $\rho\propto\alpha$. Also ρ is a monotonically increasing function of a as in the $d=1$ random walk case. The same argument applies to the alternating case (where$a\rightarrow -a$). This explains the linear dependence of ρ on α for small α in Fig. 4 and the qualitative a dependence of the slope at $\alpha=0$.

In the limit $\alpha\rightarrow 2\pi$, we expect $\rho\rightarrow\alpha/2\pi$ since the '''mean-free path'' of the particle between samplings is very much greater than the width of the absorbing region, so that the particle does not '''see'' the absorbing region and, hence, the probability distribution is not affected by it . Thus doubling $(2\pi-\alpha)$ doubles the probability of being absorbed. This is true for any ΔT although since the ''mean-free path'' $\propto\sqrt{\Delta T}$, as α is decreased from 2π the linear relationship will fail sooner for smaller ΔT. The same argument applies for the alternating case, but since there are two alternating absorbing regions, the probability distribution has more time to equilibrate near each absorbing region, thus the linear dependence breaks down at smaller α for a given ΔT. This explains the linear dependence near $\alpha=2\pi$ in Fig. 4, and its breakdown as α decreases.

In the continuum limit ($a=1$) we have the result $\theta=\mu\pi/\alpha$. Since $\rho=a^{\theta/\mu}$ for $a=1$ we get $\rho=1$ for all α. The $a>0$ plots in Fig. 4 are tending to this limit for $a\rightarrow 1$. In the alternating continuum limit ($a=-1$), the particle cannot survive for $\alpha<\pi$, since the two sectors that the particle must occupy alternately are disjoint. For $\alpha>\pi$ these two sectors overlap, and it is clear that the alternating persistence problem ($a=-1$) is equivalent to the usual persistence ($a=1$) but with α replaced by $\alpha-\pi$, i.e., $\theta=\pi\mu/(\alpha-\pi)$. So $\rho=0$ for $\alpha<\pi$ and $\rho=1$ for $\alpha>\pi$, i.e., we get a step function at $\alpha=\pi$. The $a<0$ plots are tending to this limit for $a\rightarrow -1$.

From Eq. (45) it can be seen that $\rho(1/a)=a^2\rho(a)$ for $a<1$, which enables one to construct the results for a in the range $a<-1$ from those in the range $-1<a<0$. The extra factor of $1/a$ compared to the $d=1$ case is due to the phase space factor rdr rather than dx in the eigenvalue integral equation.

V. RANDOMLY ACCELERATED PARTICLE

We now consider the nonMarkovian random-acceleration problem. We first recast the equation $\ddot{x}=\eta(t)$ as the two first-order equations $\dot{v}=\eta(t)$ and $\dot{x}=v$. After the change of variables $V=v/t^{3/2}$, $X=x/t^{1/2}$, $T=\ln t$ these equations become

$$dV/dT=-\alpha V+\eta(T), \tag{48}$$

$$dX/dT=-\beta X+V, \tag{49}$$

with $\alpha=1/2$ and $\beta=3/2$ for the random-acceleration problem, the process being ''smooth'' for any α and β. The noise correlator is $\langle\eta(T)\eta(T')\rangle=2D\delta(T-T')$. By eliminating V, Eqs. (48) and (49) can be written as

$$d^2X/dT^2+(\alpha+\beta)dX/dT+\alpha\beta X=\eta(T), \tag{50}$$

which is the equation for an overdamped simple-harmonic oscillator, $\alpha=\beta$ corresponding to critical damping. Note that Eq. (50) is symmetric under $\alpha\leftrightarrow\beta$. Letting $T\rightarrow T/(\alpha+\beta)$ and $X\rightarrow X/(\alpha+\beta)^{3/2}$ gives

$$\ddot{X}+\dot{X}+\frac{\alpha/\beta}{(1+\alpha/\beta)^2}X=\eta(T), \tag{51}$$

showing that $\theta=\alpha f(\alpha/\beta)$ and so only differing ratios of α to β need be considered.

For continuum persistence, Sinai [35] and also Burkhardt [2] have shown that $\theta=1/4$ for the random-acceleration problem ($\alpha=1/2$, $\beta=3/2$). For ratios other than $\beta=3\alpha$, no analytic solutions have been found, except for the case $\alpha/\beta\ll 1$ that we give in this paper.

To derive our usual eigenvalue integral equation, we need to find the propagator $P(X,V,T+\Delta T|Y,U,T)$, that is, the probability of going from (Y, U) at time T to (X, V) at time $T+\Delta T$. For a Gaussian process, finding the means and variances will completely specify the distribution.

Let $\tilde{X}=X-\langle X\rangle,\tilde{V}=V-\langle V\rangle$ and define $A=e^{-\alpha\Delta T}$, $B=e^{-\beta\Delta T}$. Then the propagator is

$$P(X,V|Y,U)=\frac{1}{2\pi\sqrt{\mathrm{Det}\,M}}\exp\left[-\frac{1}{2}(\tilde{X},\tilde{V})M^{-1}(\tilde{X},\tilde{V})\right], \tag{52}$$

where

$$M=\begin{pmatrix}\langle\tilde{X}^2\rangle & \langle\tilde{X}\tilde{V}\rangle\\ \langle\tilde{X}\tilde{V}\rangle & \langle\tilde{V}^2\rangle\end{pmatrix}$$

and

$$\tilde{X}=X-YB-\frac{U}{\beta-\alpha}(A-B), \tag{53}$$

$$\tilde{V}=V-UA, \tag{54}$$

$$\langle\tilde{X}^2\rangle=\frac{1}{\alpha\beta(\beta-\alpha)^2(\beta+\alpha)}[(\beta-\alpha)^2-\beta(\beta+\alpha)A^2+4\alpha\beta AB-\alpha(\beta+\alpha)B^2], \tag{55}$$

$$\langle\tilde{X}\tilde{V}\rangle=\frac{1}{\alpha(\beta-\alpha)(\beta+\alpha)}[(\beta-\alpha)-(\beta+\alpha)A^2+2\alpha AB], \tag{56}$$

$$\langle\tilde{V}^2\rangle=\frac{1}{\alpha}(1-A^2), \tag{57}$$

and the eigenvalue integral equation is

$$\rho P_\infty(X,V)=\int_0^\infty dY\int_{-\infty}^\infty dU P_\infty(Y,U)P(X,V|Y,U), \tag{58}$$

where we are working in the forward variable $P_\infty(X,V)$, the probability of finding the particle at position X with velocity

V given that it has been found at positive X at all previous samplings. Making the change of variable, $X/\sqrt{2\,\mathrm{Det}\,M}=x$, $V/\sqrt{2\,\mathrm{Det}\,M}=v$, $Y/\sqrt{2\,\mathrm{Det}\,M}=y$, $U/\sqrt{2\,\mathrm{Det}\,M}=u$, $P_\infty(X,V)=f(x,v)$ gives

$$\rho f(x,v)=\frac{1}{\pi}\sqrt{\mathrm{Det}\,M}\int_0^\infty dy\int_{-\infty}^\infty du f(y,u)\exp[-(\tilde{x}^2\langle\tilde{V}^2\rangle -2\tilde{x}\tilde{v}\langle\tilde{X}\tilde{V}\rangle+\tilde{v}^2\langle\tilde{X}^2\rangle)]. \tag{59}$$

Substituting

$$\tilde{v}=v-Au,$$

$$\tilde{x}=x-yB-u\frac{A-B}{\beta-\alpha}, \tag{60}$$

into Eq. (59), the exponent will contain all pair combinations of x, v, y, u. This kernel is not symmetrizable in $x\leftrightarrow y$ with $v\leftrightarrow u$, hence, the variational method is not applicable. Expanding in the four mixed terms, xy, xu, vy, vu, and using the same method as before, we obtain a matrix equation of the form

$$\rho I_{c,d}=\mathbf{M}_{c,d,e,f}I_{e,f} \tag{61}$$

(see Appendix A). As previously, we have found ρ numerically and also as a series expansion in $a=\exp(-\Delta T/2)$. The results for $\rho(a)$ are presented in Fig. 5, and the corresponding results for $\theta(a)$ in Fig. 6. The coefficients of the expansion up to order 25 are given in Appendix B for the case $\alpha=1/2$, $\beta=3/2$. As for the random walk case, the series have not yet converged for $a\to 1$. Since we now have a four-dimensional matrix, the problem is more apparent since we have only been able to reach $O(a^{25})$. We extend the curves by use of Padé Approximants [38], which relies on $\rho(a)$ being "smooth," and so do not work for the random walk for $a\to 1$ because of the $\sqrt{\Delta T}$ cusp. Also, (see Fig. 5), the Padé method does not work in the alternating case for large β because of the sharp downturn for $a\to -1$. Nevertheless, in the remaining cases it significantly extends the valid range for ρ, and we use this for plotting the remaining alternating persistence cases. However, since $\theta=\frac{1}{2}\ln(\rho)/\ln(a)$, when $a\to 1$ the $1/\ln(a)$ accentuates the slight error in ρ. To remedy this, we add an extra term to the Padé polynomial in the numerator (or denominator) to enforce the constraint $\rho(1)=1$. Figure 5 shows plots of the eigenvalue against $\exp(-\Delta T/2)$ for $\alpha=1/2$ and various values of β. Also plotted are the alternating persistence results.

Surprisingly, for the continuum limit of alternating persistence $\rho\not\to 0$. Indeed, $\rho\approx 0.108$ for all α, β. Previously, we had thought [30] that nonzero ρ could only occur for a "rough" process (which has an infinite number of crossings in finite time). The reason for these results is as follows. For alternating persistence, X (and similarly V) must cross zero inside a time ΔT so for $\Delta T\to 0$, X,V must both tend to zero. As a result, Eqs. (48) and (49) become $\dot{V}(T)=\eta(T)$, $\dot{X}(T)=V(T)$, or equivalently $\ddot{X}=\eta(T)$, thus removing the α, β dependence. This latter equation is invariant under changes

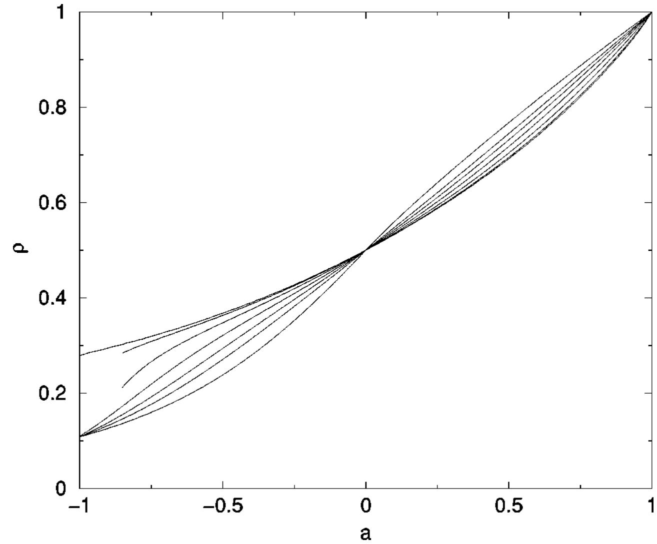

FIG. 5. Plot of $\rho(a)$, $a=e^{-\alpha\Delta T}$, $\alpha=1/2$ in all cases, with $a<0$ denoting alternating persistence. For $a>0$ the curves are, from the bottom: random walk ($\beta=\infty$) to $O(a^{49})$, and constrained Padés $\beta=24$, $\beta=6$, $\beta=3$, $\beta=2$, $\beta=3/2$, $\beta=1$ to $O(a^{25})$. For $a<0$ the ordering is reversed, the random walk is $O(a^{49})$ and the random accelerations are $O(a^{19})$. The Padé curves shown are the averages of the diagonal Padé (for odd orders) and the four nearest off-diagonal Padés, although these do not differ visibly [except for the plots of $\theta(a)$ for a larger than about 0.99, see Fig. 6]. Only Padés that do not have spurious poles on the real axis in the range $-0.25<a<1.25$ are considered. Note that for β finite, $\rho(-1)\approx 0.108$ while for the random walk $\rho(-1)=0.280\ldots$. As $\beta\to\infty$, the turn down occurs closer to $a=-1$ and is sharper, and so for $\beta=6,24$, the Padé can no longer "predict" the curves and in these two cases only the raw series are plotted.

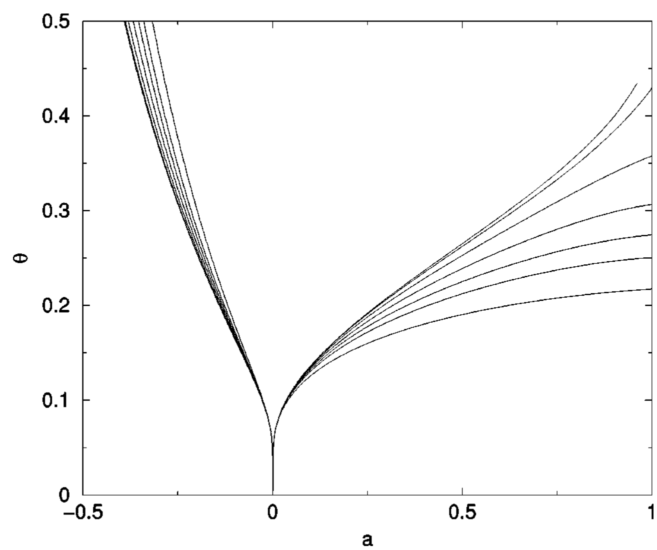

FIG. 6. Plot of $\theta(a)[=\frac{1}{2}\ln(\rho)/\ln(a)]$, $a=e^{-\alpha\Delta T}$, $\alpha=1/2$ in all cases, where ρ is the same as for Fig. 5. $a<0$ denotes alternating persistence. For $a>0$ the curves are, from the top: random walk ($\beta=\infty$), $\beta=24$, $\beta=6$, $\beta=3$, $\beta=2$, $\beta=3/2$, $\beta=1$. For $a<0$, the order is reversed. Note that the random walk series has not converged for $a\to 1$, but is 1/2 in this limit.

in T and hence ΔT, provided we rescale X. It, therefore, gives a nonzero ρ even for $\Delta T \to 0$. This implies that any process whose equation is (or becomes, for $\Delta T \to 0$) time-scale invariant has $\rho_{alt} \neq 0$ for $\Delta T \to 0$. For example, the diffusion equation from random initial conditions is equivalent to the $n \to \infty$ limit of the process $d^n x/dt^n = \eta(t)$ and will reduce to $d^n X/dT^n = \eta(T)$ for $\Delta T \to 0$.

We can compare our results to exact results for two limiting cases. For $\Delta T \to \infty$, as before we expect $\rho = \frac{1}{2} + \frac{1}{\pi} \arcsin[C(\Delta T)]$, where for this case the normalized correlation function is

$$C(\Delta T) = \frac{\beta e^{-\alpha \Delta T} - \alpha e^{-\beta \Delta T}}{\beta - \alpha}. \tag{62}$$

Hence,

$$\rho = 1/2 + \frac{1}{\pi} \frac{\beta}{\beta - \alpha} e^{-\alpha \Delta T} + O(e^{-2\alpha \Delta T}, e^{-\beta \Delta T})$$

for $\beta > \alpha$. For $\alpha/\beta \to 0$, the normalized correlator, Eq. (62), becomes $C(\Delta T) = \exp(-\alpha \Delta T)$, the random walk correlator. In fact, Eq. (50) reduces to the random walk equation, $\dot{X} = -\alpha X + \eta(T)$, if the limit $\alpha \ll \beta$ is taken after the change of variables $X = \chi/\beta\sqrt{\alpha}$, $T = \tau/\alpha$, when it becomes clear that the inertial term is negligible for $\alpha/\beta \to 0$. For $\alpha/\beta \ll 1$, we can also solve for θ in the continuum limit, $\Delta T = 0$, by expanding around the Markov process $\dot{X} = -\alpha X + \eta(T)$ to first order in $\sqrt{\alpha/\beta}$ [11,12,36]. The correlator $C(T)$ can be written as

$$C(T) = e^{-\alpha T} + \frac{\alpha}{\beta - \alpha}(e^{-\alpha T} - e^{-\beta T}). \tag{63}$$

Using the standard perturbation expansion [36] for θ for a process with correlator $C(T) = e^{-\alpha T} + \epsilon a(T)$, with $\epsilon \ll 1$,

$$\theta = \alpha \left(1 - \frac{2\alpha}{\pi} \frac{\alpha}{\beta - \alpha} \int_0^\infty dT \frac{a(T)}{(1 - e^{-2\alpha T})^{3/2}} \right) + O(\epsilon^2), \tag{64}$$

gives

$$\theta = \alpha \left(1 - \frac{2}{\sqrt{\pi}} \frac{\alpha}{\beta - \alpha} \Gamma\left[\frac{\beta}{2\alpha} \right] / \Gamma\left[\frac{\beta - \alpha}{2\alpha} \right] \right). \tag{65}$$

To first order in $\sqrt{\alpha/\beta}$, therefore,

$$\theta = \alpha \left(1 - \sqrt{\frac{2}{\pi}} \sqrt{\frac{\alpha}{\beta}} \right) + O\left(\frac{\alpha}{\beta} \right). \tag{66}$$

For $\alpha = 1/2$, $\beta = 24$, this gives $\theta = 0.442 + O(\alpha/\beta)$, which is consistent with the value of $\theta = 0.429$ obtained from the constrained Padé.

The series for $\rho(a)$ in powers of a agree with those found using the correlator expansion [34] for the various values of α and β tried. This was used as a powerful check of the accuracy of the correlator expansion.

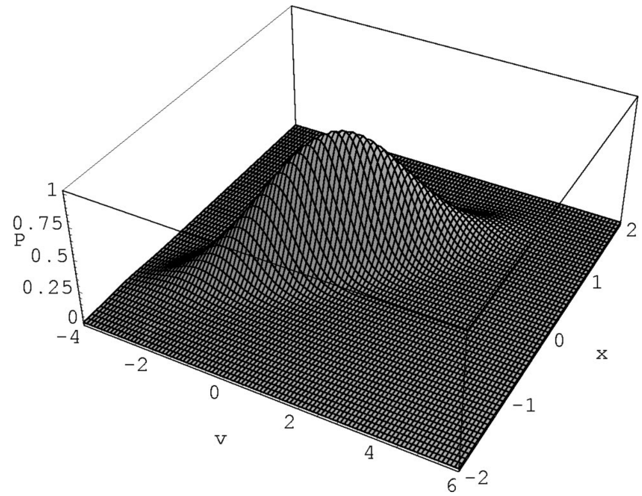

FIG. 7. The eigenfunction for $\alpha = 1/2$, $\beta = 3/2$, and $a = 0.1$.

As before, the eigenfunction $P_\infty(X,V)$ can be reconstructed from the eigenvector corresponding to the largest eigenvalue using Eq. (A4). The result is displayed in Fig. 7 for the standard case $\alpha = 1/2$ and $\beta = 3/2$.

VI. UNDERDAMPED NOISY SIMPLE-HARMONIC OSCILLATOR

We now consider the underdamped case of the noisy simple-harmonic oscillator (SHO). The persistence exponent for this case is not known even in the continuum limit. Furthermore the correlator of the process is oscillatory, which to our knowledge has not yet been studied. From Eq. (50) it can be seen that the complex values

$$\alpha = \gamma + i\omega,$$

$$\beta = \gamma - i\omega, \tag{67}$$

correspond to an underdamped oscillator. The corresponding equation of motion is

$$\ddot{X} + 2\gamma \dot{X} + (\gamma^2 + \omega^2) X = \eta(T), \tag{68}$$

where the angular frequency of oscillation is ω and the decay rate is γ. Substituting α and β into the correlator, Eq. (62), gives

$$C(T) = \exp(-\gamma T) \left[\cos(\omega T) + \frac{\gamma}{\omega} \sin(\omega T) \right]. \tag{69}$$

Since α and β are complex, we choose to study this process using the correlator expansion about $\Delta T \to \infty$, that was developed in Ref. [34]. We merely substitute the correlator, Eq. (69), into our 14th order series expansion for $\rho(\Delta T)$ and hence find θ_D. Note that for the random walk and random-acceleration problems above, 14th order corresponds to order a^{14}. As before, $\theta_D = \gamma f(\omega/\gamma)$ so we choose to keep $\gamma = 1/2$ and vary ω. Figure 8 shows plots of θ against a for various values of ω. Also shown are the random walk and persistence and alternating persistence exponents, which are nu-

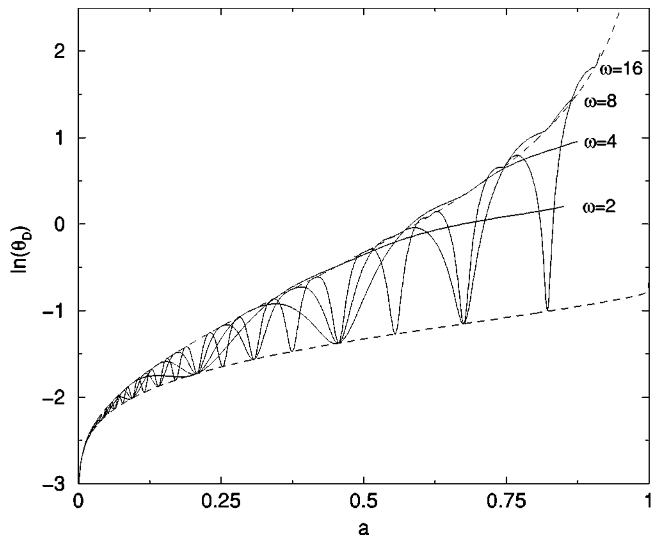

FIG. 8. Plots of the underdamped noisy SHO persistence exponent against $a=e^{-\Delta T/2}$, calculated using the correlator expansion for the cases $\gamma=1/2$ with $\omega=2,4,8,16$ from the lower right, respectively. Also plotted are the discrete random walk (lowest-dashed curve) and alternating discrete random walk (upper-dashed curve) that are equal to the oscillating exponents for $\omega\Delta T=2m\pi$ and $\omega\Delta T=(2m+1)\pi$, respectively, where m is an integer. Note that the series have not converged for $a\to 1$.

merically identical to the underdamped SHO when $\omega\Delta T=2m\pi$ and $\omega\Delta T=(2m+1)\pi$, respectively, with m as an integer. Note that, as before, the series have not converged sufficiently for small ΔT.

The (at first sight surprising) identity of the random walk and underdamped SHO persistence exponents for certain values of ΔT is interesting, and the explanation rather simple: Since these are Gaussian processes of zero mean, their correlators possess all the information about them. When studying discrete-sampling persistence, we are in effect studying a sequence of random variables, $X_1,X_2,X_3,\ldots,X_n$, whose correlator we know since $\langle X_p X_q\rangle=C(|p-q|\Delta T)$. Any processes that have identical correlators for all $C(m\Delta T)$, m integer, will have identical discrete persistence exponents for this value of ΔT. From Eq. (69) it can be seen that for $\omega\Delta T=2m\pi$ the correlator reduces to the random walk correlator, $\exp(-\gamma\Delta T)$, hence, the agreement of the persistence exponents. For $\omega\Delta T=(2m+1)\pi$, $C(m\Delta T)=\exp(-\gamma\Delta T)(-1)^m$, that is, the correlator alternates in sign. This corresponds to alternating persistence since $X(m\Delta T)$ has in effect changed sign for m odd.

At this point we make a general comment about the variation of discrete persistence exponents with ΔT. Consider a process for which we know $\theta(\Delta T)$ for some range of ΔT, e.g., from using the matrix method or correlator expansion for small ΔT. If we change ΔT to $\Delta T/m$, m integer, we will sample at exactly the same times as before, plus the intermediate times. Hence, some paths that were persistent before will be excluded, and so the asymptotic survival probability decreases. Thus,

$$\theta_D(\Delta T/m)\geqslant\theta_D(\Delta T). \tag{70}$$

This of course does not imply that $\theta_D(\Delta T)$ is monotonic. Taking the limit $m\to\infty$, and assuming that $\theta_D(\Delta T)$ is smooth for $\Delta T\to 0$, we get the result that

$$\theta\geqslant\theta_D(\Delta T)\quad\forall\Delta T. \tag{71}$$

This provides a lower bound on θ for when our large ΔT expansions do not converge up to the continuum limit. We can use this to see that, for $\gamma=1/2$, $\theta\geqslant 1.22$, 2.60, 4.49, and 7.19, $\omega=2$, 4, 8, and 16, respectively.

VII. CONCLUSION

In this paper we have extended our earlier treatment of the discrete persistence exponent for the random walk with an absorbing boundary at the origin [30] to the two-dimensional random walk with absorbing boundaries on a wedge, and the random-acceleration process with an absorbing boundary at the origin. While the latter is perhaps the simplest continuous-time non-Markovian process, both the processes discussed in this work have the simplifying feature that they can be written as a Markov process in two dimensions. We have shown that, in both processes, the discrete persistence probability after n measurements, $Q_n(x)$, for a process starting at x, has the asymptotic form $Q_n(x)\sim\rho^n q(x)$, where ρ and $q(x)$ are the largest eigenvalue and corresponding eigenfunction of a certain eigenvalue integral equation. They have been evaluated to high precision by converting the integral equation into a matrix eigenvalue problem, from which high-order series expansions in powers of $a=\exp(-\mu\Delta T)$ have been obtained, where ΔT is a uniform measurement interval for a Gaussian stationary process (GSP). The random walk and random-acceleration problems have been mapped onto GSPs in logarithmic time $T=\ln t$, and for these (and similar) processes the calculations presented here apply to the case of measurements uniformly spaced in T.

The case of alternating persistence, in which the measured values of the stochastic variable take positive and negative values alternately, has been discussed using a formal continuation of a to negative values. The case $a>1$ (or $a<-1$ for alternating persistence), corresponds, for a GSP, to motion in an unstable potential. For $a>1$ there is a nonzero probability $q(x)$ that the process, starting at $x>0$, is never measured to be negative, and we have shown how to calculate it, with explicit results (see, e.g., Fig 3) for the Ornstein-Uhlenbeck process.

For the random-acceleration process, the corresponding GSP is a noisy, overdamped harmonic oscillator $\ddot{X}+(\alpha+\beta)\dot{X}+\alpha\beta X=\eta(T)$ with $\alpha=1/2$ and $\beta=3/2$. This process is clearly of interest for other values of α and β, since even the value of θ in the continuum limit ($\Delta T\to 0$) is not known exactly except for $\beta=3\alpha$. We have obtained a perturbative result for the continuum limit in the limit $\alpha\ll\beta$. We have also investigated the underdamped case, corresponding to complex α and β, and shown that for discrete measurements with time step ΔT equal to an integer number of oscillation periods, the persistence properties are identical to those of a random walk, while for an odd half-integer number one recovers the alternating persistence of a random walk.

The methods presented here become increasingly unwieldy as the order of the stochastic differential equation increases. Recently, a power series approach has been developed in which the eigenvalue ρ is expanded in powers of the correlator $C(k\Delta T)$ evaluated at integer multiples k, of the time step between measurements, with the maximum value of k depending on the order of the expansion [34]. While not as powerful as the matrix method for systems described by low-order stochastic differential equations, the series approach has the advantage that it can be applied to any GSP, including those (such as diffusion from random initial conditions) that cannot be described by a differential equation of finite order.

The "simple" persistence problem discussed here deals with the probability of detecting no zero crossing in n measurements. The statistics of the number of zero crossings, i.e., the probability to observe m crossings in n measurements, is also of interest. Results obtained by applying both the methods of this paper and the series expansion approach of Ref. [34] will be reported in a separate publication.

ACKNOWLEDGMENT

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APPENDIX A: CALCULATION OF RANDOM-ACCELERATION MATRIX EQUATION

Let the xv coefficient in the exponent be a_{xv}, etc., then

$$a_{xx}=-\langle\tilde{V}^2\rangle,$$

$$a_{xv}=2\langle\tilde{X}\tilde{V}\rangle,$$

$$a_{vv}=-\langle\tilde{X}^2\rangle,$$

$$a_{yy}=-B^2\langle\tilde{V}^2\rangle,$$

$$a_{yu}=-2B\frac{A-B}{\beta-\alpha}\langle\tilde{V}^2\rangle+2AB\langle\tilde{X}\tilde{V}\rangle,$$

$$a_{uu}=-A^2\langle\tilde{X}^2\rangle-\left(\frac{A-B}{\beta-\alpha}\right)^2\langle\tilde{V}^2\rangle+2A\frac{A-B}{\beta-\alpha}\langle\tilde{X}\tilde{V}\rangle,$$

$$a_{xy}=2B\langle\tilde{V}^2\rangle,$$

$$a_{vy}=-2B\langle\tilde{X}\tilde{V}\rangle,$$

$$a_{xu}=2\frac{A-B}{\beta-\alpha}\langle\tilde{V}^2\rangle-2A\langle\tilde{X}\tilde{V}\rangle,$$

$$a_{vu}=2A\langle\tilde{X}^2\rangle-2\frac{A-B}{\beta-\alpha}\langle\tilde{X}\tilde{V}\rangle. \tag{A1}$$

Expanding the exponent of Eq. (59) in powers of terms that mix (x,v) with (y,u), we get,

$$\rho f(x,v)=\frac{1}{\pi}\sqrt{\text{Det}\,M}\int_0^\infty dy\int_{-\infty}^\infty du f(y,u)\times\exp[a_{yy}y^2+a_{yu}yu+a_{uu}u^2]\times\sum_{n=0}^\infty\frac{1}{n!}(a_{xy}xy+a_{vy}vy+a_{xu}xu+a_{vu}vu)^n\times\exp[a_{xx}x^2+a_{xv}xv+a_{vv}v^2]. \tag{A2}$$

Letting

$$I_{c,d}=\int_0^\infty dx\int_{-\infty}^\infty dv f(x,v)x^c v^d\times\exp[a_{yy}x^2+a_{yu}xv+a_{uu}v^2] \tag{A3}$$

gives

$$\rho f(x,v)=\frac{\sqrt{\text{Det}\,M}}{\pi}\sum_{e=0}^\infty\sum_{f=0}^\infty\frac{I_{ef}}{(e+f)!}\binom{e+f}{e}\sum_{r=0}^{e}\sum_{s=0}^{f}\binom{e}{r}\times\binom{f}{s}a_{xy}^{e-r}a_{vy}^{r}a_{xu}^{f-s}a_{vu}^{s}x^{e-r+f-s}v^{r+s}\times\exp[a_{xx}x^2+a_{xv}xv+a_{vv}v^2]. \tag{A4}$$

Multiplying through by $x^c v^d$ and integrating over $x>0$ and over all v, gives

$$\rho I_{c,d}=\frac{1}{\pi}\sqrt{\text{Det}M}\,G_{c,d,e,f}I_{e,f}, \tag{A5}$$

where

$$G_{c,d,e,f}=\frac{1}{(e+f)!}\binom{e+f}{e}\sum_{r=0}^{e}\sum_{s=0}^{f}\binom{e}{r}\times\binom{f}{s}a_{xy}^{e-r}a_{vy}^{r}a_{xu}^{f-s}a_{vu}^{s}D_{e-r+f-s+c,r+s+d} \tag{A6}$$

and

$$D_{a,b}=\int_0^\infty dx\int_{-\infty}^\infty dv\; x^a v^b\exp[-(\mathcal{A}v^2-\mathcal{B}xv+\mathcal{C}x^2)], \tag{A7}$$

giving

$$D_{a,b}=\sqrt{\frac{\pi}{2}}\sum_{t=0}^{[b/2]}\binom{b}{b-2t}(2\mathcal{A})^{-\frac{(2t+1)}{2}}\left(\frac{\mathcal{B}}{2\mathcal{A}}\right)^{b-2t}\times\left(\mathcal{C}-\frac{\mathcal{B}^2}{4\mathcal{A}}\right)^{-(\frac{a+b-2t+1}{2})}(2t-1)!!\times\Gamma\left(\frac{a+b-2t+1}{2}\right), \tag{A8}$$

where $[b/2]$ indicates the integer part of $b/2$ and

$$\mathcal{A}=-a_{vv}-a_{uu}, \tag{A9}$$

$$\mathcal{B}=a_{xv}+a_{yu}, \tag{A10}$$

$$\mathcal{C}=-a_{xx}-a_{yy}. \tag{A11}$$

Thus the problem has been reduced to computing the largest eigenvalue of an $N\times N\times N\times N$ operator. As in the wedge case, $G_{c,d,e,f}$ decouples into $G_{(c,d),(e,f)}$. For alternating persistence, the range of integration over y in Eq. (A2) should be from $-\infty$ to 0. Substituting $y\to -y$ and $u\to -u$, and changing the limits of integration gives

$$\rho f(x,v)=\frac{\sqrt{\mathrm{Det}\,M}}{\pi}\int_0^\infty dy\int_{-\infty}^\infty du f(-y,-u)\sum_{n=0}^\infty \frac{1}{n!}\times\exp[a_{yy}y^2+a_{yu}yu+a_{uu}u^2]\times(-a_{xy}xy-a_{vy}vy-a_{xu}xu-a_{vu}vu)^n\times\exp[a_{xx}x^2+a_{xv}xv+a_{vv}v^2]. \tag{A12}$$

Due to the symmetry of the system, $f(y,u)=f(-y,-u)$, so changing $a_{ij}\to -a_{ij}(i=x,v,j=y,u)$ in the matrix method gives alternating persistence.

APPENDIX B: SERIES FOR ρ FOR RANDOM WALK AND RANDOM ACCELERATION

The coefficients for the random walk are

$$\begin{aligned}\rho(a)=&0.500\,000+0.318\,310a^1+0.115\,668a^2+0.021\,446a^3+0.015\,651a^4+0.015\,762a^5+0.000\,050a^6-0.003\,320a^7\\&+0.007\,597a^8+0.007\,587a^9-0.004\,372a^{10}-0.005\,964a^{11}+0.004\,840a^{12}+0.007\,913a^{13}-0.001\,290a^{14}\\&-0.006\,184a^{15}-0.000\,369a^{16}+0.004\,205a^{17}+0.001\,467a^{18}-0.000\,757a^{19}+0.000\,989a^{20}+0.000\,385a^{21}\\&-0.003\,272a^{22}-0.002\,408a^{23}+0.003\,354a^{24}+0.004\,972a^{25}-0.000\,072a^{26}-0.003\,737a^{27}-0.001\,786a^{28}\\&+0.000\,650a^{29}+0.000\,270a^{30}+0.000\,415a^{31}+0.002\,188a^{32}+0.001\,629a^{33}-0.001\,600a^{34}-0.002\,448a^{35}\\&+0.000\,213a^{36}+0.001\,451a^{37}-0.000\,501a^{38}-0.001\,372a^{39}+0.000\,740a^{40}+0.002\,125a^{41}+0.000\,743a^{42}\\&-0.000\,602a^{43}-0.000\,379a^{44}-0.000\,732a^{45}-0.001\,956a^{46}-0.001\,143a^{47}+0.001\,780a^{48}+0.002\,824a^{49}+O(a^{50}).\end{aligned}$$

The coefficients for the random-acceleration are

$$\begin{aligned}\rho(a)=&0.500\,000+0.477\,464a+0.021\,519a^2+0.000\,314a^3+0.035\,886a^4-0.063\,298a^5+0.029\,548a^6+0.032\,884a^7\\&-0.061\,472a^8+0.020\,790a^9+0.030\,977a^{10}-0.039\,237a^{11}+0.016\,107a^{12}+0.012\,402a^{13}-0.035\,066a^{14}\\&+0.021\,396a^{15}+0.020\,271a^{16}-0.027\,422a^{17}+0.002\,006a^{18}+0.005\,649a^{19}-0.009\,557a^{20}+0.018\,307a^{21}\\&+0.002\,099a^{22}-0.023\,960a^{23}+0.007\,071a^{24}+0.012\,405a^{25}+O(a^{26}).\end{aligned}$$

The coefficients in both series have been truncated to six decimal places for brevity. In the plots presented in the paper, sufficient precision has been retained in the coefficients to ensure the accuracy of the plots and of any quoted values for ρ and θ.

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